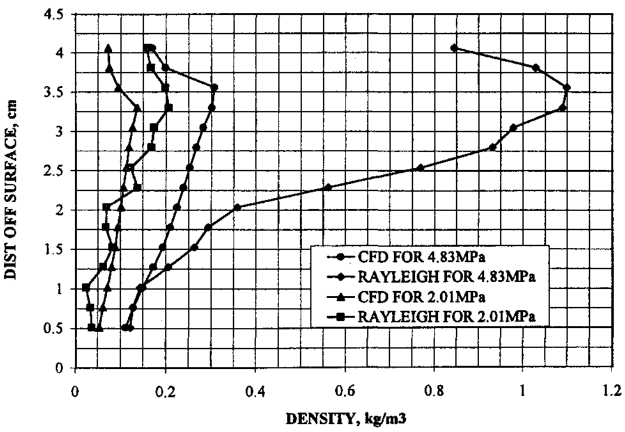

Fig. 4 Comparison of Rayleigh scattering measurements and CFD results.

Rayleigh scatter measurements and CFD results. Close to the surface of the cone the measurements agree quite well, while farther off the surface behind the shock wave and into the freestream it is apparent that the Rayleigh scatter measurements are substantially higher than expected. Although the quantitative results disagree, it is reassuring to see that qualitatively the two techniques, Rayleigh scattering and CFD, appear similar.

Conclusions

Extraneous surface scatter and scatter off condensation particles create difficulties in using Rayleigh scattering as a measurement technique in hypersonic flows. Fortunately, the surface scatter background noise can be eliminated by taking scattered intensity, readings at two known density conditions, obtaining the linear relation between density and scattered intensity, and calculating the level of extraneous surface scatter. Further work will be performed toward reduction and possible elimination of condensation effects hindering Rayleigh scattering measurement efforts.

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Vibration and Stability of Simply Supported Elliptical Plates

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Introduction

ELLIPTICAL plates are widely used as cover plates for cutouts in structural components. The precise determination of frequencies and critical compressive loads of elliptical plates involves considerable difficulties in the integration of the fourth-order partial differential equation. The fundamental frequency parameters of elliptical plates with clamped and simply supported end conditions are obtained by using different techniques.[1,2] The elastic stability of a circular plate under compressive force N uniformly distributed around the edge of the plate has been extensively investigated.[3] For the case of simply supported circular plates, the critical load parameter $\lambda_b (\equiv Na^2/D)$ is given by[3]

$$\lambda_b = \beta^2 \tag{1}$$

where β is the smallest root of the characteristic equation,

$$\beta J_0(\beta) - (1-\nu) J_1(\beta) = 0$$

and where a is the radius of the circular plate, D is the flexural rigidity, J_0 and J_1 are zeroth- and first-order Bessel's functions, respectively, and ν is the Poisson's ratio. Reference 4 provides the details on the stability of a clamped elliptical plate, which was investigated in 1937 by Woinowsky-Krieger using the energy method. An approximate calculation for the stability of simply supported elliptical plates was also suggested based on the results for circular plates. Clamping the edges increases the critical stress by a factor of 3.5 in circular plates and by a factor of 4.0 in rectangular plates.

For an elliptical plate the factor should lie between 3.5 and 4, depending on the eccentricity of the ellipse. Hence, the values of the critical stress for a clamped elliptical plate should be divided by a factor of between 3.5 and 4 to obtain the critical stress for a simply supported elliptical plate. This criterion may yield the lower and upper bound solutions for the stability of simply supported elliptical plates.

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Jones[5] and Sundararajan[6] have discussed approximate solutions for the fundamental frequency of plates through static deflections. Motivated by the work of these investigators, we extend the problem here to the stability analysis of plates.

Formulation

Consider a linear elastic plate occupying an area A, inside the boundary S, undergoing free harmonic vibration. According to the Rayleigh–Ritz method, the fundamental frequency ω can be obtained by selecting a function $\psi(x, y)$ for the lateral deflection w of the plate that would satisfy the boundary conditions and then minimizing the resultant potential and kinetic energies. As a first approximation, the function ψ can be selected as an expression that is proportional to a static deflection with the same conditions of edge fixing and subjected to uniformly distributed load. This is equivalent to the assumption that the plate surface, which corresponds to the fundamental mode, is identical to that deflected by a uniformly distributed load.

The expression for the possible deflection is taken in the form

$$w = W_{\max}\,\psi(x, y) \tag{2}$$

where $W_{\max}$ is the maximum deflection of the plate under uniformly distributed load q and

$$|\psi(x, y)| \leq 1, \forall (x, y) \in S$$

Using Eq. (2), one can write expressions for potential energy U and the work done W as

$$U = \tfrac{1}{2} D W_{\max}^2 U_\psi \tag{3}$$

$$W = \int\!\!\int_A q W_{\max}\psi(x, y)\,\mathrm{d}x\,\mathrm{d}y = q W_{\max} W_\psi \tag{4}$$

where

$$U_\psi = \int\!\!\int_A \left[\left(\frac{\partial^2\psi}{\partial x^2}\right)^2 + \left(\frac{\partial^2\psi}{\partial y^2}\right)^2 + 2\nu\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\psi}{\partial y^2} + 2(1-\nu)\left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2\right]\mathrm{d}x\,\mathrm{d}y$$

Minimizing the total potential energy ($\equiv U - W$) with respect to $W_{\max}$, one obtains

$$D U_\psi W_{\max} = q W_\psi \tag{5}$$

The lateral deflection of a plate undergoing free harmonic vibration having frequency ω can be expressed as

$$w = \phi(t)\psi(x, y) \tag{6}$$

Then the expressions for potential energy U and kinetic energy T become

$$U = \tfrac{1}{2} D \phi^2 U_\psi \tag{7}$$

$$T = \frac{1}{2}\rho h\omega^2 \int\!\!\int_A (\phi\psi)^2\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{2}\rho h\omega^2\phi^2 T_\psi \tag{8}$$

where ρ is the mass per unit volume and h is the thickness of the plate. Minimizing the resultant potential and kinetic energies with respect to ϕ, one gets

$$D U_\psi = \rho h\omega^2 T_\psi \tag{9}$$

From Eqs. (5) and (9), a relation between ω and $W_{\max}$ is obtained as

$$\rho h\omega^2 (W_{\max}/q) = \alpha_f \tag{10}$$

where

$$\alpha_f = \frac{W_\psi}{T_\psi} = \int\!\!\int_A \psi\,\mathrm{d}x\,\mathrm{d}y \bigg/ \int\!\!\int_A \psi^2\,\mathrm{d}x\,\mathrm{d}y$$

Extension of the present method to the buckling analysis of plates yields

$$D U_\psi = \int\!\!\int_A \left[N_{xb}\left(\frac{\partial\psi}{\partial x}\right)^2 + N_{yb}\left(\frac{\partial\psi}{\partial y}\right)^2 + 2N_{xyb}\left(\frac{\partial\psi}{\partial x}\right)\left(\frac{\partial\psi}{\partial y}\right)\right]\mathrm{d}x\,\mathrm{d}y \tag{11}$$

where N_{xb}, N_{yb}, and N_{xyb} are the applied compressive stress resultants. From Eqs. (5) and (11), a relation between the compressive stress resultants and $W_{\max}$ is obtained as

$$\left(\frac{W_{\max}}{q}\right)\int\!\!\int_A \left[N_{xb}\left(\frac{\partial\psi}{\partial x}\right)^2 + N_{yb}\left(\frac{\partial\psi}{\partial y}\right)^2 + 2N_{xyb}\left(\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial y}\right)\right]\mathrm{d}x\,\mathrm{d}y = W_\psi \tag{12}$$

Method of Solution for a Simply Supported Elliptical Plate

The boundary of the elliptical plate is expressed as

$$(x/a)^2 + (y/b)^2 = 1 \tag{13}$$

where a and b are semimajor and semiminor axes of the ellipse, respectively. Equation (12) for an elliptical plate under a compressive force N uniformly distributed around the edge of the plate becomes

$$\left(\frac{W_{\max}}{q}\right) N \int\!\!\int_A \left(\frac{x}{a^2}\frac{\partial\psi}{\partial x} + \frac{y}{b^2}\frac{\partial\psi}{\partial y}\right)^2 \times \left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right)^{(-1)}\mathrm{d}x\,\mathrm{d}y = W_\psi \tag{14}$$

A three-term deflection function ψ for w that satisfies the geometric boundary conditions of zero edge deflection and possesses twofold symmetry is chosen as

$$w = W_{\max}\psi(x, y) = W_{\max}\left(1 + \delta_1\xi^2 + \delta_2\eta^2\right)(\xi^2 + \eta^2 - 1) \tag{15}$$

where $\xi = x/a$ and $\eta = y/b$.

Substituting $\psi(x, y)$ in Eqs. (3) and (4) and minimizing the total potential energy ($\equiv U - W$), one gets

$$[a_{ij}]\{X_i\} = -(qb^4/D)\{b_i\} \tag{16}$$

where $[a_{ij}]$ is a 3×3 symmetric matrix having elements

$$a_{11} = (2K^4 + 4\nu K^2 + 2), \qquad a_{12} = (3K^4 + 4\nu K^2 + 1)/2$$

$$a_{13} = (K^4 + 4\nu K^2 + 3)/2$$

$$a_{22} = [63K^4 + (18\nu + 8)K^2 + 3]/12$$

$$a_{23} = [3K^4 + (6\nu + 8)K^2 + 3]/12$$

$$a_{33} = [3K^4 + (18\nu + 8)K^2 + 63]/12, \qquad K = b/a$$

where $\{X_i\}$ is a 3×1 column matrix having elements

$$X_1 = W_{\max}, \qquad X_2 = W_{\max}\delta_1, \qquad X_3 = W_{\max}\delta_2$$

and where $\{b_i\}$ is a 3×1 column matrix having elements

$$b_1 = \tfrac{1}{4}, \qquad b_2 = \tfrac{1}{24}, \qquad b_3 = \tfrac{1}{24}$$

The maximum deflection $W_{\max}$ and the constants δ_1 and δ_2 are obtained by solving Eq. (16). The constant α_f in Eq. (10) becomes

$$\alpha_f = \frac{20(6 + \delta_1 + \delta_2)}{80 + 20(\delta_1 + \delta_2) + 3\left(\delta_1^2 + \delta_2^2\right) + 2\delta_1\delta_2} \tag{17}$$

Equation (14) becomes

$$N(W_{\max}/q) = b^2\alpha_b \tag{18}$$

where

$$\alpha_b = \frac{4(6+\delta_1+\delta_2)}{16(K^2+1)(3+\delta_1+\delta_2)+(11K^2-5)\delta_1^2+3(K^2+1)\delta_2^2+2(K^2+9)\delta_1\delta_2+2(\delta_1-\delta_2)^2(K^4-2K^2+4)} \tag{19}$$

The maximum deflection $W_{\max}$, the fundamental frequency ω, and the critical compressive force N can be expressed in nondimensional form as

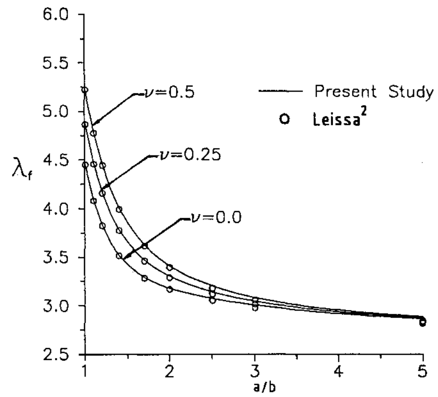
$$W_{\max} = \alpha(qb^4/D) \tag{20}$$

$$\lambda_f = \sqrt{\rho h/D}\,\omega b^2 = \sqrt{\alpha_f/\alpha} \tag{21}$$

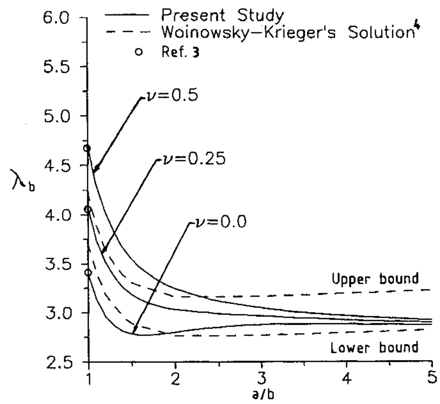
$$\lambda_b = Nb^2/D = \alpha_b/\alpha \tag{22}$$

Table 1 Comparison of the parameter α for the maximum deflection of a simply supported elliptical plate under uniformly distributed load q

	$a/b = 1$		$a/b = 2$	
ν	Ref. 7	Present study	Ref. 7	Present study
0.00	0.0781	0.0781	0.1542	0.1542
0.25	0.0656	0.0656	0.1441	0.1439
0.50	0.0573	0.0573	0.1356	0.1356



Fundamental frequency parameter λ_f



Critical load parameter λ_b

Fig. 1 Comparision of λ_f and λ_b for a simply supported elliptical plate.

Discussion of the Results

The adequacy of the present method of computing the fundamental frequency and the critical compressive force for a simply supported elliptical plate from its static deflections is examined. Table 1 gives the comparison of the maximum deflection of a simply supported elliptical plate under uniformly distributed load. The solution obtained assuming a three-term deflection function for the lateral deflection w is found to be in good agreement with those values of Ref. 7. Figure 1 shows the comparison of the fundamental frequency parameter λ_f and the critical load parameter λ_b for different values of the Poisson's ratio ν. The free vibration analysis results compare well with those of Leissa.[2] Results of the critical load parameter λ_b for a simply supported circular plate are found to be in good agreement with those obtained from Eq. (1). Woinowsky-Krieger's lower and upper bound solutions are obtained by dividing the values of the critical force for the clamped elliptical plate[4] with factors 3.5 and 4. The fundamental frequency parameter λ_f and critical compressive force parameter λ_b can be represented in the form

$$\lambda_f = 3.00543(K^4 + 2\nu K^2 + 1)^{\frac{1}{2}} \tag{23}$$

$$\lambda_b = \frac{3.12168(K^4 + 2\nu K^2 + 1)}{K^2 + 1} \tag{24}$$

which are reliable and useful in design.

Conclusions

This study confirms the applicability of static deflections of elastic plate for direct evaluation of the fundamental frequency and the critical compressive force.

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